# Comparison of Various Drag Coefficient Expansions Using Polynomials and Splines

Sally M. Sabot* and G.L. Winchenbach†
*Air Force Armament Laboratory, Eglin Air Force Base, Florida*
and
G.T. Chapman‡
*NASA Ames Research Center, Moffett Field, California*

**The longitudinal differential equation of motion has been used to investigate various aerodynamic expansion techniques. The total drag coefficient was expanded using conventional polynomials and splines with and without floating knot locations. This paper discusses the various techniques and approaches, compares results obtained from simultaneously fitting four separate flights (time vs distance measurements) and outlines the potential advantages and disadvantages of the various aerodynamic expansion techniques. It is believed that this is the first time that the knot locations associated with the various spline segments were assumed as unknown during the aerodynamic coefficient estimation process and that these results and techniques are germane to other applications.**

## Nomenclature

$A$ = reference area
$a$ = coefficient in Eq. (3)
$C_i$ = slopes of spline segments; see Eqs. (4-6)
$C_D$ = total drag coefficient
$C_{D_0}$ = zero angle of attack drag coefficient
$C_{D_2}, C_{D_4}$ = second- and fourth-order drag terms, respectively; see Eqs. (2) and (3)
$C_{D_V}$ = drag variation due to velocity change
$e$ = exponential
$m$ = model mass
$n$ = number of straight line segments; see Eq. (4)
RSQ = sum of residuals squared
$S_i$ = switches for spline segments; see Eqs. (4-6)
$V$ = velocity along the $X$ axis
$V_i, V_{\text{ref}}$ = instantaneous and reference velocities, respectively
$X$ = downrange axis
$\alpha_i, \beta_i$ = instantaneous pitch and yaw angles, respectively
$\delta$ = total instantaneous angle of attack
$\delta_i$ = knot locations in Eqs. (4-6)
$\rho$ = air density

*Superscripts*

$(\dot{\ })$ = time derivative
$(\ddot{\ })$ = second derivative with respect to time

## Introduction

PRIOR to 1969, the prevalent method of analyzing ballistic spark range data was based on the linear approximation method known as "linear theory" developed by Murphy[1,2] and others.[3-5] Stated briefly, the method uses a closed-form solution to the differential equations of motion. This approximate solution results from assuming a linearized aerodynamic model where the aerodynamic force and moment derivatives are constant with angle of attack (hence the name "linear theory"). Murphy extended this technique to include a quasinonlinear analysis,[6] where the linear aerodynamic force and momentum derivatives are reduced such that certain nonlinearities could be obtained. This quasinonlinear analysis requires an assumed functional form of the nonlinearity (normally a quadratic or cubic polynomial).

In 1969 Chapman and Kirk, in analyzing free-flight data,[7] documented the application of a technique they called parametric differentiation, which permitted the free-flight differential equation of motion to be used directly in the data correlation process. This technique eliminated the requirement for closed-form soutions to the equations of motion. However, it is still required to assume a form of the nonlinearities in the equations of motion. Generally these forms have also been assumed to be polynomial expansions of the aerodynamic force and moment derivatives with angle of attack.[8,9]

During the past several years, data analysts have discussed the possibility of using mathematical splines (two or more mathematical expressions attached end to end) for the coefficient expansions.[10] These splines permit the slopes of the aerodynamic expansions to be discontinuous and would offer the analyst a more general aerodynamic model, thereby relieving some of the requirements of assuming the form of the nonlinearities. This paper discusses various coefficient expansion techniques and compares results obtained using the various expansions.

## Method of Approach

In order to evaluate the various expansion techniques (continuous function vs splines), we will restrict our attention to a simple single-degree-of-freedom system rather than the full six-degree-of-freedom system described in various references.[8,9]

### Longitudinal Momentum Equation

This paper will examine the determination of the total drag coefficient ($C_D$) as a function of instantaneous angle of attack and velocity from the longitudinal momentum equation and the associated experimental measurements of distance

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*Cooperative Engineering Study, Auburn University.
†Chief Aeroballistics Section, Aeromechanics Division. Member AIAA.
‡Senior Research Scientist, Thermo and Gas Dynamics Division. Member AIAA.

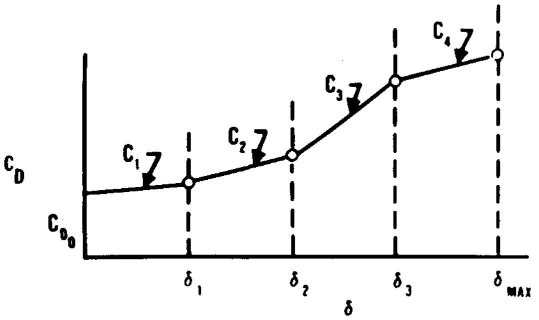


Fig. 1 General multidimensional straight line segment expansion.

traveled vs time. The differential equation governing the longitudinal momentum is

$$\dot{V}=\ddot{X}=-\frac{\rho A}{2m}\dot{X}^2\{C_D(\delta,V)\} \tag{1}$$

Equation (1) assumes that the angle between the velocity vector and the $X$ axis is small.

### Expansions for the Total Drag Coefficient

Several expansion techniques for $C_D$ were investigated. The first two involved continuous functions, beginning with the classical quadratic dependence on angle of attack, or

$$C_D=C_{D_0}+C_{D_2}\delta^2 \tag{2}$$

Equation (2) represents the classical expression for which Murphy[1,2] developed a methodology of determining $C_{D0}$ and $C_{D2}$ by plotting the effective measured drag coefficients vs the mean of $\delta^2$. A straight line through these data yields the intercept $C_{D0}$ and the slope $C_{D2}$. The second continuous function is much more versatile and is valid for a wider range of angle of attack dependence and a linear velocity dependence, that is,

$$C_D=C_{D_0}e^{a\delta}+C_{D_2}\delta^2+C_{D_4}\delta^4+C_{D_v}(\dot{X}-V_{\text{ref}}) \tag{3}$$

Note that this expansion has five unknown coefficients, $C_{D0}$, $a$, $C_{D2}$, $C_{D4}$, and $C_{D_v}$ as do all the remaining expansion techniques considered within this paper. The $e^{a\delta}$ term is somewhat unconventional but allows a nonzero slope at zero angle of attack. The $C_{D_v}$ coefficient is normally small but, for high drag configurations that experience large velocity decays during the flight or when time-distance data obtained from several flights (slightly different launch velocities) are simultaneously analyzed, this term can be important. This $C_{D_v}$ term accounts for variations in drag coefficients with Mach number and Reynolds number. The two effects cannot be easily separated because they both depend linearly on velocity. This term appears in all the expansions discussed except for the classical quadratic dependence shown in Eq. (2).

Several expansions using spline functions were also evaluated. The first of these uses multiple straight line segments (see Fig. 1) and the general expansion is

$$C_D=C_{D_0}+\sum_{i=1}^{n}\Big\{S_iC_i(\delta-\delta_i)+C_i(\delta_{i+1}-\delta_i)\sum_{j=i+1}^{n+1}S_j\Big\}+C_{D_v}(\dot{X}-V_{\text{ref}}) \tag{4}$$

The switches $S_i$ are either zero or one depending on whether the instantaneous value of $\delta$ is within the range of the segment (i.e., if $\delta<\delta_1$, then $S_1=1$ and all other switches are zero or if $\delta_1\leq\delta<\delta_2$, then $S_2=1$ and the other switches are zero, etc.). For the investigation discussed, the evaluations of the various expansion techniques were restricted to four unknown coefficients plus the $C_{D_v}$ term. Hence, only two cases utilizing straight line segments were considered. The first case is three segments ($n=3$) with the knots fixed at $\delta_1$ and $\delta_2$. These knot locations were chosen by dividing the $\delta$ range into three equal parts, $\delta_1=\delta_{\max}/3$, $\delta_2=2\delta_{\max}/3$, $\delta_3=\delta_{\max}$. The five unknown coefficients then become $C_{D0}$, $C_1$, $C_2$, $C_3$, and $C_{D_v}$. Here it should be noted that other methods of dividing the $\delta$ space into segments and selecting the knot locations were considered.

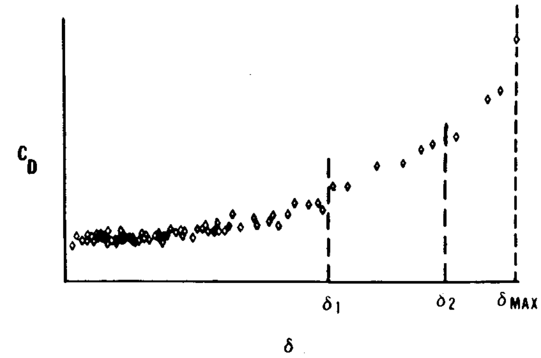


Fig. 2 Typical free-flight experimental data with smaller segments at the higher angles of attack.

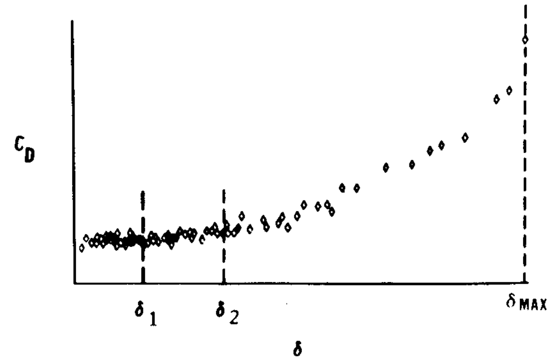


Fig. 3 Typical free-flight experimental data with equal number of data points in each segment.

Initially, it was felt that the segments should be chosen such that those associated with the higher angles of attack would be small compared to the segments associated with the smaller angles of attack. The reasoning for this was that it was assumed that the rate of change of the drag coefficient with respect to $\delta$ was much higher at large angles of attack, thereby requiring smaller segments. Although this assumption is certainly true for most free-flight range data, the nature of a well-behaved, dynamically stable configuration in free flight is that the larger initial angles of attack rapidly decrease (damp) during the flight. Hence, only relatively few data points representing the initial high angles of attack are normally obtained, compared with the number of data points associated with the smaller angles of attack (for example, see Fig. 2). Considering this, if a small segment were chosen for the higher angles of attack, only very few data points would fall within this segment, thereby invalidating the resultant slope parameter. After this anomaly was recognized, it was felt that perhaps the knot locations should be selected such that each segment contained an equal number of data points. However, it immediately became obvious that due to the nature of the data, a very small segment resulted at the smaller angles of attack, where it wasn't needed, and a large segment resulted at the large angles of attack where a small segment was desired (see Fig. 3). With these considerations in mind, dividing the $\delta$ range into equal parts appeared to be a reasonable compromise. However, if this technique is applied to other free-flight data (i.e., dynamically unstable configurations) or to another application altogether, the logic associated with selecting the knot locations should be revisited.

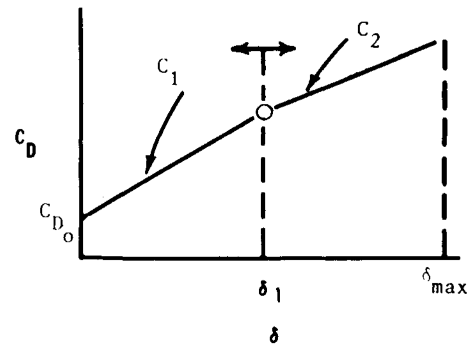


Fig. 4 Two straight line segments with floating knot location.

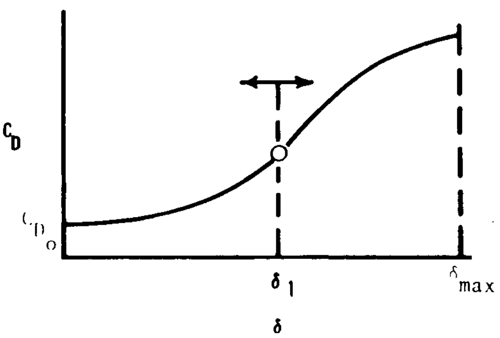


Fig. 5 Two quadratic segments with floating knot location.

The second case utilizing straight line segments splined together possessed only two segments, $n=2$ in Eq. (4), but the knot ($\delta_1$) was allowed to be a free variable and determined by the data reduction procedure. Hence, the five unknown coefficients become $C_{D0}$, $C_1$, $C_2$, $\delta_1$, and $C_{D_v}$ (see Fig. 4).

Another set of splines used in the present investigation involves quadratic segments similar to Eq. (2). Here, the knot locations are free variables and are determined by the data reduction routine (see Fig. 5). This expansion can be defined such that not only is the $C_D$ function continuous at the knot but the derivative also can be made continuous at the knot. Both of these approaches are discussed. First, consider the approach where only $C_D$ is continuous at the floating knot location. This expansion is written as

$$C_D = C_{D_0} + S_1 C_1 \delta^2 + S_2 C_1 \delta_1^2 + S_2 C_2 (\delta^2 - \delta_1^2) + C_{D_v}(\dot{X} - V_{\text{ref}}) \tag{5}$$

Here $C_{D0}$, $C_1$, $C_2$, $\delta_1$, and $C_{D_v}$ are the five free variables and the function has a discontinuous slope at $\delta_1$. $S_1$ and $S_2$ are the determining switches for the polynomials and are set similarly to those in Eq. (4) (i.e., if $\delta^2 < \delta_1^2$ then $S_1 = 1$ and $S_2 = 0$ or if $\delta^2 > \delta_1^2$, then $S_1 = 0$ and $S_2 = 1$).

Equation (5) can be modified by adding an additional term such that the slope of the $C_D$ vs $\delta$ curve is also continuous at the floating knot location to yield

$$C_D = C_{D_0} + S_1 C_1 \delta^2 + S_2 C_1 \delta_1^2 + C_{D_v}(\dot{X} - V_{\text{ref}}) + 2S_2(C_1 - C_2)(\delta - \delta_1)\delta_1 + S_2 C_2(\delta^2 - \delta_1^2) \tag{6}$$

The unknown coefficients for this expression are the same as for Eq. (5), and the switches are set similarly.

There are many other spline expansions that could be considered using five free unknown coefficients; however, it is believed that the ones defined herein are sufficient to illustrate the applicability and usefulness of these techniques.

#### Parameter Identification

The unknown free coefficients in the various expansions, Eqs. (2-6), are determined by fitting the experimentally

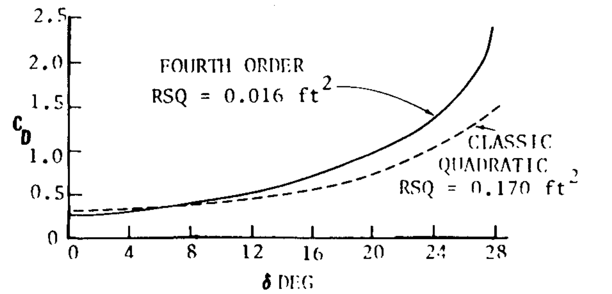


Fig. 6 Comparison of classic quadratic and fourth-order expansions.

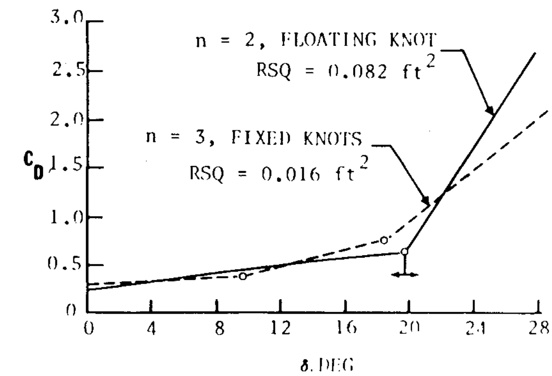


Fig. 7 Comparison of two multiple straight segment fits [see Eq. (4)].

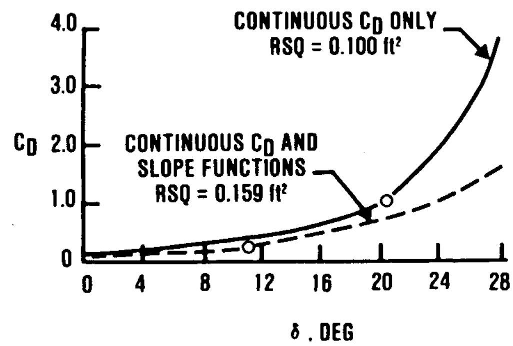


Fig. 8 Comparison of quadratic splines with continuous and discontinuous slopes.

measured time and distance data with the numerical solution of Eq. (1). This fitting process is a least-squares technique, and the angle-of-attack history is provided as an input. The method used is the one described by Chapman and Kirk,[7] which parametrically differentiates the equation of motion with respect to each of the unknown coefficients. Numerical integration of the equation of motion and parametric equations is then used to match the theoretical equation of motion to the experimental measurements. It should be noted that when the knot locations are assumed to be unknown free coefficients [Eqs. (5) and (6)], they are treated as parameters in exactly the same way as are the aerodynamic coefficients. Basically the only difference in this fitting process from that described in Ref. 7 is the logic required to keep track of when the switches need to be set.

## Results and Discussion

The estimated total drag coefficient ($C_D$) variations obtained using the various expansion techniques, Eqs. (2-6), are shown in Figs. 6-8. These figures not only graphically depict the values of the assumed functional form of $C_D$ but also show the sum of the residuals squared (RSQ) achieved during the fitting process of each function. The RSQ is a statistical

**Table 1 Effect of various polynomial terms**

| | Cases | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| $C_{D_0}$ | 0.31 | 0.31 | 0.31 | 0.31 | 0.31 |
| $C_{D_2}$ | 5.35 | 5.22 | 3.89 | 5.05 | −1.17 |
| $C_{D_4}$ | – | – | 14.68 | – | 35.18 |
| $a$ | – | – | – | 0.14 | 2.25 |
| $C_{D_v} \times 10^3$ | – | −0.09 | −0.11 | −0.09 | −0.12 |
| RSQ, ft$^2$ | 0.17 | 0.10 | 0.07 | 0.10 | 0.02 |

measure of how well the mathematical expression represents the experimentally measured data (time and distance). Each of the $C_D$ expansions was evaluated by simultaneously fitting four sets of experimentally measured time and distance data obtained from four separate flights of a 25-mm spin stabilized projectile tested in the Aeroballistics Research Facility.[11] This set of data was used to evaluate the $C_D$ expansions, because of the relatively high angles of attack experienced during some of the flights and the apparent highly nonlinear characteristics of $C_D$ with total angle of attack. The initial velocity (muzzle) varied from 3168 to 3245 ft/s for the four flights, the average midrange velocity ($V_{REF}$) of all four flights being 3098 ft/s.

Here it should be cautioned that because some of the expansion techniques fit this particular set of data better than others, it doesn't necessarily mean that one expansion method is superior to another. In fact, the analyst should recognize that when selecting an expansion (whether or not it is one of the expansions discussed herein) for a particular application the inherent nature of the data should be the dominate consideration. For example, if time and position data obtained from the flight of a sphere were being analyzed (i.e., angle of attack is of no concern), the expansion would only include the $C_{D0}$ and $C_{D_v}$ terms. Or, if a highly nonlinear spring-mass-damper system were being analyzed, the frequency of oscillation and/or damping might be modeled as a function of the displacement using multiple straight line segments with or without floating knots. For this case, one may possess information that identifies where the knot should be located. Also, a good rule of thumb is to use the simplest expansion that adequately matches that particular set of data. With these considerations in mind, the $C_D$ expansions using the various techniques previously discussed are presented to show the applicability and versatility of these methods. Furthermore, they graphically illustrate to any potential user that the more conventional continuous functions are not the only choices available.

Figure 6 shows the comparison of the classical quadratic [Eq. (2)] with the fourth-order polynomial including both the $e^{a\delta}$ and $C_{D_v}$ terms [Eq. (3)]. As shown in this figure, the fourth-order polynomial resulted in a significantly better fit to the four separate flights than the quadratic (note the RSQ for both expansions). However, this should not be unexpected, since the fourth-order polynomial has three additional unknown coefficients ($a$, $C_{D4}$, and $C_{D_v}$), and this added flexibility would be expected to yield a lower RSQ. The real question here is whether the dramatic rise in $C_D$ at the highest angles of attack is real. The four flights used for these comparisons had a total of 139 data points, of which only four data points possessed angles of attack greater than 20 deg. Hence, the added flexibility may be providing erroneous results at the higher angles of attack. This would be especially true if one of the four data points, about 20 deg, was in error. Normally, because of the paucity of data at the higher angles of attack, the $C_D$ expansions would only be presented up to about 20 deg. But, for the purpose of this paper, the expansions are shown up to the maximum angle of attack.

It is also of interest to see the effect of the various terms in the fourth-order polynomial, Eq. (3). Table 1 presents the results obtained from running five different cases with various terms held at zero in the fourth-order polynomial. In this table, case 1 represents the classical quadratic as plotted in Fig. 6. Case 5 represents the complete fourth-order polynomial expansion, also shown in Fig. 6. Cases 2-4 represent the results obtained when the remaining three unknown coefficients, $C_{Dv}$, $C_{D4}$, and $a$, are included in the reduction routine. When viewing the results shown in Table 1, it is apparent that the $C_{Dv}$ term is important for this particular set of data (note RSQ for case 2 compared with case 1) and appears to be consistently determined for all cases. What is remarkable about the results shown in Table 1 is the dramatic improvement in the overall quality of fit (lower RSQ) when all of the unknown coefficients are determined simultaneously (see case 5). This indicates that this particular set of time-position data requires a relatively complex expansion of $C_D$ with angle of attack and velocity.

The comparison of the three-segment linear spline, Eq. (4) with $n=3$, containing fixed knot locations to the two-segment linear spline [also Eq. (4) with $n=2$] containing a floating knot location is shown in Fig. 7. This figure illustrates that the fit containing three segments is superior to the two segments fitted with the floating knot (note the RSQ for both fits). This result is not unexpected since the quality of fit should improve by increasing the number of segments. What is interesting is that the sum of the residual squared for the three-segment case is the same as the fourth-order polynomial shown in Fig. 6 (RSQ = .016 ft$^2$ for both). It is suspected that this is a result of the fortuitous knot location (arbitrarily set by dividing the $\delta$ space into three equal segments) at 18.3 deg.

The quadratic splines with a continuous $C_D$ expansion only [Eq. (5)] and a continuous $C_D$ and slope expansion [Eq. (6)] are compared in Fig. 8. Neither of these expansions resulted in a fit to the chosen set of data as well as the fourth-order polynomial or the three-segment linear spline. Both of these expansions also contain a condition in which the coefficient extraction technique becomes indeterminate. This condition exists when the user permits the knot location to be a free variable, and the slope of the first segment is nearly equal to the slope of the second segment ($C_1 \approx C_2$). When this occurs, the minimum RSQ is insensitive to the location of the knot and the iteration process fails to converge. However, if it is recognized that this condition exists, the knot location can be held constant and the iterative process again becomes stable. This condition is also indicative of an overly defined system. For instance, a single quadratic expansion would fit the data as well as the two quadratic expansions splined together. It seems that, for the set of data used, neither quadratic expansion appears to be advantageous. Nevertheless, this techniques does appear promising for modeling other aerodynamic coefficients that are highly nonlinear such as Magnus terms.

## Conclusions

Various total drag coefficient expansions using polynomials and/or splines have been developed and compared. A set of four flights was selected to show the applicability and versatility of these methods.

All the expansions provided a better fit to the experimental data than the classical linear theory, suggesting the need for a relatively complex expansion of $C_D$ with angle of attack and velocity. The fourth-order polynomial and the three-segment spline provided superior fits. However, these results may be unique for this particular data set and do not imply that they are the best expansions to use in every case.

These techniques, as applied in the aerodynamic coefficient estimation process, show great potential for other highly nonlinear applications. Moreover, they demonstrate that conventional, continuous functions are not the only choices available to the data analyst.

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